

Local a Priori Estimate on the General Scale Subdomains*

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Abstract

The local a priori estimate for the finite element approximation is essential for underlying the local and parallel technique. It is well known that the constant coefficients in the inequality is independent of the mesh size. But it is not so clear whether the constant depends on the scale of the local subdomains. The aim of this note is to derive a new local a priori estimate on the general scale domains. We also show that the dependence of the constant appearing in the local a priori estimate on the scale of the subdomains.

Keywords. Scaling argument, scale of subdomain, local priori error estimate, parallel computation.

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1 Introduction

Recently, parallel techniques for the finite element computation have become very attractive. There exists a type of parallel schemes which are based on the understanding of the local and global properties of a finite element solution for the elliptic type problems which is proposed in [6] and then has been studied extensively [3, 4, 7, 8]. The cornerstone of this technique is the local a priori estimate [5, 6], where the involved constant is independent of the mesh parameters. However, the dependence of the constant on the scale of the subdomains is not so clear. If we want to consider the sharp effects caused by local subdomain (Ω_0), it is necessary to clarify the dependence of the coefficient on the diameter of Ω_0 (denoted by d_{Ω_0}).

*This work is supported in part by the National Science Foundation of China (NSFC 11001259, 11371026, 11201501, 11031006, 2011CB309703), the National Center for Mathematics and Interdisciplinary Science, CAS and the President Foundation of AMSS-CAS.

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For example, in [3, 7, 8], we need to know how large of the subdomain to construct the efficient parallel method. In this paper, we explicitly show the dependence of the local priori error estimate on the subdomain scale d_{Ω_0} .

The rest of this paper is organized as follow. In the next section, some notation, assumptions and basic results are listed. In Section 3, a local estimate of the finite element solution is derived on the general scale domain and the dependence of the local estimates on the subdomain scale is clarified. Some concluding remarks are given in the last section.

2 Preliminaries

In this section, following [6], we firstly state the model problem and list some basic notations and results. Then we set some reasonable assumptions on the finite element spaces and show their reasonability.

2.1 Model problem

Let Ω be a bounded domain in \mathcal{R}^d ($d \geq 1$). We shall use standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and seminorms [1]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace, and $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$. In some places, $\|\cdot\|_{s,2,\Omega}$ should be viewed as piecewise defined if it is necessary. For $D \subset G \subset \Omega$, the notation $D \subset\subset G$ means that $\text{dist}(\partial D \setminus \partial\Omega, \partial G \setminus \partial\Omega) > 0$. Note that any $w \in H_0^1(\Omega_0)$ can be naturally extended to be a function in $H_0^1(\Omega)$ with zero outside of Ω_0 . Thus we will state this fact by the abused notation $H_0^1(\Omega_0) \subset H_0^1(\Omega)$.

In this paper, we mainly consider the following second order elliptic problem:

$$\begin{cases} \mathcal{L}u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Here \mathcal{L} is a general linear second order elliptic operator:

$$\mathcal{L}u = - \sum_{i,j=1}^d \frac{\partial}{\partial x_j} \left(a_{ij} \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} + \phi u$$

with $a_{ij}, b_i \in W^{1,\infty}(\Omega)$, $0 \leq \phi \in L^\infty(\Omega)$ and the matrix $(a_{ij})_{1 \leq i,j \leq d}$ being uniformly positive definite on Ω .

The weak form of (2.1) is as follows:

Find $u \equiv \mathcal{L}^{-1}f \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \quad (2.2)$$

where (\cdot, \cdot) is the standard inner-product of $L^2(\Omega)$ and

$$a(u, v) = a_0(u, v) + N(u, v)$$

with

$$a_0(u, v) = \int_{\Omega} \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} d\Omega \quad \text{and} \quad N(u, v) = \int_{\Omega} \left(\sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} v + \phi uv \right) d\Omega. \quad (2.3)$$

In this paper, we assume there exists constants C independent of Ω such that the follow inequalities hold

$$\|w\|_{1,\Omega} \leq C a_0(w, w), \quad \forall w \in H_0^1(\Omega), \quad (2.4)$$

and

$$a_0(u, v) \leq C \|u\|_{1,\Omega} \|v\|_{1,\Omega}, \quad N(u, v) \leq C \|u\|_{0,\Omega} \|v\|_{1,\Omega}, \quad \forall u, v \in H_0^1(\Omega). \quad (2.5)$$

In order to define higher derivatives of functions with multi variables, we introduce the following multi-index $\alpha = (\alpha_1, \dots, \alpha_d)$ and the corresponding differential operator:

$$D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \dots \partial_{x_d}^{\alpha_d}. \quad (2.6)$$

Furthermore, we say $\alpha \leq \beta$ if and only if $\alpha_i \leq \beta_i$, $i = 1, \dots, d$. And when $\alpha \geq \beta$, we denote $\alpha - \beta = (\alpha_1 - \beta_1, \dots, \alpha_d - \beta_d)$. For derivative of the product of two functions, we have

$$D^\alpha(fg) = \sum_{i=0}^{|\alpha|} \sum_{|\beta|=i, \beta+\gamma=\alpha} D^\beta f D^\gamma g, \quad (2.7)$$

where $|\alpha| = \alpha_1 + \dots + \alpha_d$.

2.2 Some assumptions on the finite element spaces

Following [6], we present some assumptions on the finite element spaces and then define the corresponding finite element approximation for the problem (2.2).

First we generate a shape-regular decomposition $\mathcal{T}_h(\Omega)$ for the computing domain $\Omega \subset \mathcal{R}^d$ ($d = 2, 3$) into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$). The diameter of a cell $K \in \mathcal{T}_h(\Omega)$ is denoted by h_K . The mesh size function is denoted by $h(x)$ whose value is the diameter h_K of the element K including x .

Now, we state the following assumption for the mesh considered in this paper:
A.0. *There exists $\gamma > 1$ such that*

$$h_\Omega^\gamma \leq C h(x), \quad \forall x \in \Omega, \quad (2.8)$$

where $h_\Omega = \max_{x \in \Omega} h(x)$ is the largest mesh size of $\mathcal{T}_h(\Omega)$ and C is a constant independent of Ω and $h(x)$.

Based on the triangulation $\mathcal{T}_h(\Omega)$, we define the finite element space $S_h(\Omega) \subset H^1(\Omega)$ and $S_h^0(\Omega) = S_h(\Omega) \cap H_0^1(\Omega)$. Given $G \subset \Omega$, we use $S_h(G)$ and $\mathcal{T}_h(G)$ to denote the restriction of $S_h(\Omega)$ and $\mathcal{T}_h(\Omega)$ to G , respectively, and define

$$S_h^0(G) = \{v \in S_h(\Omega) : \text{supp } v \subset\subset G\}. \quad (2.9)$$

For any concerned subdomain $G \subset \Omega$ in this paper, we assume that it aligns with the partition $\mathcal{T}_h(\Omega)$.

Now, we would like to state some assumptions on the finite element space. The constants C appeared here and after are independent of the scale of Ω and mesh parameters.

A.1. (*Approximation*). For any $w \in H_0^1(\Omega)$, we have

$$\inf_{v \in S_h^0(\Omega)} (\|h^{-1}(w - v)\|_{0,\Omega} + \|w - v\|_{1,\Omega}) = o(1), \quad (2.10)$$

as $h_\Omega \rightarrow 0$.

A.1'. (*Approximation*). There exists $r \geq 1$ such that for any $w \in H_0^1(\Omega)$,

$$\inf_{v \in S_h^0(\Omega)} (h_\Omega^{-1} \|w - v\|_{0,\Omega} + \|w - v\|_{1,\Omega}) \leq Ch_\Omega^s \|w\|_{1+s,\Omega}, \quad 0 \leq s \leq r. \quad (2.11)$$

A.2. (*Inverse Estimate*). For any $v \in S_h(\Omega_0)$,

$$\|v\|_{1,\Omega_0} \leq Ch_\Omega^{-1} \|v\|_{0,\Omega_0}. \quad (2.12)$$

A.3. (*Superapproximation*). For $G \subset \Omega_0$, let $\omega \in C_0^\infty(\Omega)$ with $\text{supp } \omega \subset\subset G$. Then for any $w \in S_h(G)$, there is $v \in S_h^0(G)$ such that

$$\|\omega w - v\|_{1,G} \leq Ch_G \|w\|_{1,G}. \quad (2.13)$$

To show the reasonability of the above assumptions, we state the normal Lagrange finite element spaces which satisfies the above assumptions, i.e.,

$$S_h(\Omega) = \{v \in C(\bar{\Omega}) : v|_K \in P_r(K), \forall K \in \mathcal{T}_h(\Omega)\}, \quad (2.14)$$

where $\mathcal{P}_r(K)$ denote the space of polynomials of degree not greater than the positive integer r .

Now, we can to investigate the new versions of Assumptions **A.1**, **A.1'**, **A.2** and **A.3** on the general subdomain scales. For this aim, we need to introduce the affine mapping which transforms the general domain Ω_0 to the reference domain $\hat{\Omega}_0$ with size 1. The affine mapping can be defined as follows:

$$F : \Omega_0 \rightarrow \hat{\Omega}_0, \quad x \rightarrow \xi := \frac{x - x_0}{d_{\Omega_0}}, \quad (2.15)$$

where x_0 is any inner point of Ω_0 . Through this map, K and $\mathcal{T}_h(\Omega_0)$ are transformed to \widehat{K} and $\widehat{\mathcal{T}}_h(\widehat{\Omega}_0)$, respectively. It is obvious that

$$\frac{h_{\Omega_0}}{d_{\Omega_0}} = \frac{h_{\widehat{\Omega}_0}}{d_{\widehat{\Omega}_0}}.$$

We define $\widehat{u}(\xi) = u(x)$ with $\xi = \frac{x-x_0}{d_{\Omega_0}}$ for $u(x)$ with $x \in \Omega_0$. Then it is naturally that $\widehat{uw} = \widehat{u}\widehat{v}$. Similarly to (2.6), we also define

$$\widehat{D}^\alpha = \partial_{\xi_1}^{\alpha_1} \partial_{\xi_2}^{\alpha_2} \cdots \partial_{\xi_d}^{\alpha_d}. \quad (2.16)$$

It is easy to derive that $\widehat{D}^\alpha \widehat{u}(\xi) = d_{\Omega_0}^{|\alpha|} D^\alpha u(x)$ and $D^\alpha u(x) = d_{\Omega_0}^{-|\alpha|} \widehat{D}^\alpha \widehat{u}(\xi)$. Then we can define the corresponding $\widehat{S}_h(\widehat{\Omega}_0)$ which can be viewed as the transformation of $S_h(\Omega_0)$ through the map (2.15).

Proposition 2.1. *If we take $S_h(\Omega)$ as in (2.14), then assumptions A.1, A.1' and A.2 hold. Assumption A.3 should be changed to the following version:*

A.3. (Superapproximation). *For $G \subset \Omega_0$, let $\omega \in C_0^\infty(\Omega)$ with $\text{supp } \omega \subset\subset G$. Then for any $w \in S_h(G)$, there is $v \in S_h^0(G)$ such that*

$$\|\omega w - v\|_{1,G} \leq C d_{\Omega_0}^{-1} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}} \right)^r \|w\|_{0,\Omega_0} + C \frac{h_{\Omega_0}}{d_{\Omega_0}} \|w\|_{1,\Omega_0}. \quad (2.17)$$

Proof. First, it is obvious that the space $S_h(\Omega)$ satisfies Assumptions **A.1**, **A.1'**, **A.2** (c.f. [2]). Here we mainly concern the proof of Assumption **A.3**. From $\omega \in C_0^\infty(\Omega_0)$, $\widehat{\omega} \in C_0^\infty(\widehat{\Omega}_0)$, Assumption **A.1'** and polynomial interpolation theory in [2], there exist $\widehat{v} \in S_h^0(\widehat{\Omega}_0)$ such that

$$|\widehat{\omega}\widehat{w} - \widehat{v}|_{1,\widehat{\Omega}_0}^2 \leq C h_{\widehat{\Omega}_0}^{2r} \left(\sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} |\widehat{\omega}\widehat{w}|_{1+r,\widehat{K}}^2 \right), \quad (2.18)$$

$$\|\widehat{\omega}\widehat{w} - \widehat{v}\|_{0,\widehat{\Omega}_0}^2 \leq C h_{\widehat{\Omega}_0}^{2+2r} \left(\sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} |\widehat{\omega}\widehat{w}|_{1+r,\widehat{K}}^2 \right), \quad (2.19)$$

where C is a constant independent of $\widehat{\Omega}_0$. From Leibnitz formula (2.7) and $\widehat{w}|_{\widehat{K}} \in \widehat{P}^r(\widehat{K})$ on any element $\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)$, the following inequalities hold

$$\begin{aligned} \sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} |\widehat{\omega}\widehat{w}|_{1+r,\widehat{K}}^2 &= \sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} \int_{\widehat{K}} \sum_{|\alpha|=1+r} |\widehat{D}^\alpha(\widehat{\omega}\widehat{w})|^2 d\widehat{K} \\ &= \sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} \int_{\widehat{K}} \sum_{|\alpha|=1+r} \left| \sum_{i=0}^{1+r} \sum_{|\beta|=i, \beta \leq \alpha, \gamma=\alpha-\beta} \widehat{D}^\beta \widehat{\omega} \widehat{D}^\gamma \widehat{w} \right|^2 d\widehat{K} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} \int_{\widehat{K}} \sum_{|\alpha|=1+r} \left| \sum_{i=0}^r \sum_{|\beta|=i, \beta \leq \alpha, \gamma=\alpha-\beta} \widehat{D}^\beta \widehat{w} \widehat{D}^\gamma \widehat{w} \right|^2 d\widehat{K} \\
&\leq \sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} \int_{\widehat{K}} \sum_{|\alpha|=1+r} C_{r,d} \sum_{i=0}^r \sum_{|\beta|=i, \beta \leq \alpha, \gamma=\alpha-\beta} |\widehat{D}^\beta \widehat{w}|^2 |\widehat{D}^\gamma \widehat{w}|^2 d\widehat{K} \\
&= \sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} \int_{\widehat{K}} C_{r,d} \sum_{i=0}^r \sum_{|\beta|=i} |\widehat{D}^\beta \widehat{w}|^2 \sum_{|\gamma+\beta|=1+r} |\widehat{D}^\gamma \widehat{w}|^2 d\widehat{K} \\
&\leq C_{r,d} \sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} \int_{\widehat{K}} \sum_{i=0}^r \sum_{|\beta|=i} |\widehat{D}^\beta \widehat{w}|^2 d\widehat{K}. \tag{2.20}
\end{aligned}$$

We take $v(x) = \widehat{v}(\xi)$ with $\xi = \frac{x-x_0}{d_{\Omega_0}}$ and claim that v is the desired function in Assumption **A.3**. In fact, by changing variables and combining (2.18) and (2.20), we have

$$\begin{aligned}
&|\omega w - v|_{1,\Omega_0}^2 = \int_{\Omega_0} |\nabla(\omega w - v)|^2 d\Omega_0 \\
&= d_{\Omega_0}^{-2} \frac{|\Omega_0|}{|\widehat{\Omega}_0|} \int_{\widehat{\Omega}_0} |\widehat{\nabla}(\widehat{\omega} \widehat{w} - \widehat{v})|^2 d\widehat{\Omega}_0 \leq C d_{\Omega_0}^{d-2} h_{\widehat{\Omega}_0}^{2r} \sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} |\widehat{\omega} \widehat{w}|_{1+r,\widehat{K}}^2 \\
&\leq C d_{\Omega_0}^{d-2} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}} d_{\widehat{\Omega}_0} \right)^{2r} C_{r,d} \sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} \int_{\widehat{K}} \sum_{i=0}^r \sum_{|\beta|=i} |\widehat{D}^\beta \widehat{w}|^2 d\widehat{K} \\
&\leq C d_{\Omega_0}^{d-2} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}} d_{\widehat{\Omega}_0} \right)^{2r} C_{r,d} \sum_{\widehat{K} \in \widehat{\mathcal{T}}_h(\widehat{\Omega}_0)} \frac{|\widehat{K}|}{|K|} \int_K \sum_{i=0}^r \sum_{|\beta|=i} d_{\Omega_0}^{2i} |D^\beta w|^2 d\widehat{K} \\
&\leq C d_{\Omega_0}^{d-2-2r} h_{\Omega_0}^{2r} C_{r,d} \max_{K \in \mathcal{T}_h(\Omega_0)} \frac{|\widehat{K}|}{|K|} \sum_{K \in \mathcal{T}_h(\Omega_0)} \int_K \sum_{i=0}^r \sum_{|\beta|=i} d_{\Omega_0}^{2i} |D^\beta w|^2 d\widehat{K} \\
&\leq C d_{\Omega_0}^{d-2-2r} h_{\Omega_0}^{2r} C_{r,d} \sum_{K \in \mathcal{T}_h(\Omega_0)} \sum_{i=0}^r d_{\Omega_0}^{2i} \left(\sum_{|\beta|=i} \|D^\beta w\|_{0,K}^2 \right). \tag{2.21}
\end{aligned}$$

Together with the inverse inequality, (2.21) can be reduced to

$$\begin{aligned}
|\omega w - v|_{1,\Omega_0}^2 &\leq C d_{\Omega_0}^{d-2-2r} h_{\Omega_0}^{2r} C_{r,d} \sum_{K \in \mathcal{T}_h(\Omega_0)} \left(\|w\|_{0,K}^2 + d_{\Omega_0}^2 |w|_{1,K}^2 + d_{\Omega_0}^4 h_{\Omega_0}^{-2} |w|_{1,K}^2 \right. \\
&\quad \left. + \cdots + d_{\Omega_0}^{2r} h_{\Omega_0}^{-2(r-1)} |w|_{1,K}^2 \right) \\
&\leq C d_{\Omega_0}^{d-2-2r} h_{\Omega_0}^{2r} C_{r,d} \sum_{K \in \mathcal{T}_h(\Omega_0)} |w|_{1,K}^2 \left(d_{\Omega_0}^2 + d_{\Omega_0}^4 h_{\Omega_0}^{-2} + \cdots \right. \\
&\quad \left. + d_{\Omega_0}^{2r} h_{\Omega_0}^{-2(r-1)} \right) + C C_{r,d} d_{\Omega_0}^{-2} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}} \right)^{2r} \|w\|_{0,\Omega_0}^2
\end{aligned}$$

$$\begin{aligned}
&\leq CC_{r,d} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}} \right)^2 \|w\|_{1,\Omega_0}^2 \left(\frac{h_{\Omega_0}^{2r-2}}{d_{\Omega_0}^{2r-2}} + \frac{h_{\Omega_0}^{2r-4}}{d_{\Omega_0}^{2r-4}} + \cdots + 1 \right) \\
&\quad + CC_{r,d} d_{\Omega_0}^{-2} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}} \right)^{2r} \|w\|_{0,\Omega_0}^2 \\
&\leq CC_{r,d} d_{\Omega_0}^{-2} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}} \right)^{2r} \|w\|_{0,\Omega_0}^2 + CC_{r,d} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}} \right)^2 \|w\|_{1,\Omega_0}^2. \tag{2.22}
\end{aligned}$$

From Poincaré inequality, we have

$$\|\omega w - v\|_{0,\Omega_0} \leq C d_{\Omega_0} |\omega w - v|_{1,\Omega_0}. \tag{2.23}$$

Then combining (2.22) and (2.23), we obtain the desired result (2.17) and the proof is complete. \square

3 Local a priori estimate

In this section, we derive a new local a priori estimate which is dependent on the subdomain scale and is different from the one in [6] where the local a priori estimate are provided for the case with the subdomain scale being $\mathcal{O}(1)$. The local estimate here is for the general subdomain scales.

The following lemma is the same as [6, Lemma 3.1]. But here we need to prove it for the general scale subdomains Ω_0 .

Lemma 3.1. *Let $D \subset\subset \Omega_0$, and let $\omega \in C_0^\infty(\Omega)$ be such that $\text{supp } \omega \subset\subset \Omega_0$. Then*

$$a_0(\omega w, \omega w) \leq 2a(w, \omega^2 w) + C \|w\|_{0,\Omega_0}^2, \quad \forall w \in H_0^1(\Omega). \tag{3.1}$$

Proof. With integration by parts, we have the following identity

$$\begin{aligned}
a_0(\omega w, \omega w) &= a(w, \omega^2 w) - N(\omega w, \omega w) + \int_{\Omega} \sum_{j=1}^d b_j \frac{\partial \omega}{\partial x_j} \omega w^2 d\Omega \\
&\quad + \int_{\Omega} \sum_{i,j=1}^d a_{ij} \left(\left(\frac{\partial \omega}{\partial x_i} \frac{\partial(\omega w)}{\partial x_j} - \frac{\partial \omega}{\partial x_j} \frac{\partial(\omega w)}{\partial x_i} \right) w + \frac{\partial \omega}{\partial x_i} \frac{\partial \omega}{\partial x_j} w^2 \right) d\Omega. \tag{3.2}
\end{aligned}$$

Let us define

$$T_1(\omega, w) = \int_{\Omega} \sum_{j=1}^d b_j \frac{\partial \omega}{\partial x_j} \omega w^2 d\Omega,$$

and

$$T_2(\omega, w) = \int_{\Omega} \sum_{i,j=1}^d a_{ij} \left(\left(\frac{\partial \omega}{\partial x_i} \frac{\partial(\omega w)}{\partial x_j} - \frac{\partial \omega}{\partial x_j} \frac{\partial(\omega w)}{\partial x_i} \right) w + \frac{\partial \omega}{\partial x_i} \frac{\partial \omega}{\partial x_j} w^2 \right) d\Omega.$$

Then we can rewrite the identity (3.2) as follows

$$a_0(\omega w, \omega w) = a(w, \omega^2 w) - N(\omega w, \omega w) + T_1(\omega, w) + T_2(\omega, w).$$

With the transform operator F defined in (2.15) and following the bilinear form (2.3), we define

$$\widehat{a}_0(u, v) = \int_{\widehat{\Omega}} \sum_{i,j=1}^d \widehat{a}_{ij} \frac{\widehat{\partial} \widehat{u}}{\partial \xi_i} \frac{\widehat{\partial} \widehat{v}}{\partial \xi_j} d\widehat{\Omega} \quad \text{and} \quad \widehat{N}(\widehat{u}, \widehat{v}) = \int_{\widehat{\Omega}} \left(\sum_{i=1}^d \widehat{b}_i \frac{\widehat{\partial} \widehat{u}}{\partial \xi_i} \widehat{v} + \widehat{\phi} \widehat{u} \widehat{v} \right) d\widehat{\Omega}. \quad (3.3)$$

Thus

$$\widehat{a}(\widehat{u}, \widehat{v}) = \widehat{a}_0(\widehat{u}, \widehat{v}) + \widehat{N}(\widehat{u}, \widehat{v}).$$

Similarly, we define

$$\begin{aligned} \widehat{T}_1(\widehat{\omega}, \widehat{w}) &= \int_{\widehat{\Omega}} \sum_{j=1}^d \widehat{b}_j \frac{\widehat{\partial} \widehat{\omega}}{\partial \xi_j} \widehat{\omega} \widehat{w}^2 d\widehat{\Omega}, \\ \widehat{T}_2(\widehat{\omega}, \widehat{w}) &= \int_{\widehat{\Omega}} \left(\sum_{i,j=1}^d \widehat{a}_{ij} \left(\frac{\widehat{\partial} \widehat{\omega}}{\partial \xi_i} \frac{\widehat{\partial}(\widehat{\omega} \widehat{w})}{\partial \xi_j} - \frac{\widehat{\partial} \widehat{\omega}}{\partial \xi_j} \frac{\widehat{\partial}(\widehat{\omega} \widehat{w})}{\partial \xi_i} \right) \widehat{w} + \frac{\widehat{\partial} \widehat{\omega}}{\partial \xi_i} \frac{\widehat{\partial} \widehat{\omega}}{\partial \xi_j} \widehat{w}^2 \right) d\widehat{\Omega}. \end{aligned}$$

Then the following identity holds:

$$\widehat{a}_0(\widehat{\omega} \widehat{w}, \widehat{\omega} \widehat{w}) = \widehat{a}(\widehat{w}, \widehat{\omega}^2 \widehat{w}) - \widehat{N}(\widehat{\omega} \widehat{w}, \widehat{\omega} \widehat{w}) + \widehat{T}_1(\widehat{\omega}, \widehat{w}) + \widehat{T}_2(\widehat{\omega}, \widehat{w}).$$

By changing variable, we have

$$\begin{aligned} a_0(\omega w, \omega w) &= d_{\Omega_0}^{-2} \frac{|\Omega_0|}{|\widehat{\Omega}_0|} \widehat{a}_0(\widehat{\omega} \widehat{w}, \widehat{\omega} \widehat{w}) \\ &\leq C d_{\Omega_0}^{d-2} (\widehat{a}(\widehat{w}, \widehat{\omega}^2 \widehat{w}) - \widehat{N}(\widehat{\omega} \widehat{w}, \widehat{\omega} \widehat{w}) + \widehat{T}_1(\widehat{\omega}, \widehat{w}) + \widehat{T}_2(\widehat{\omega}, \widehat{w})) \\ &\leq C (a(w, \omega^2 w) - N(\omega w, \omega w) + T_1(\omega, w) + T_2(\omega, w)), \end{aligned}$$

where C is a constant independent of the scale of Ω_0 . Then the rest of the proof is the same as the proof of [6, Lemma 3.1]. \square

Now, we come give the local a priori estimate for the general scale domains.

Theorem 3.1. *Suppose that $f \in H^{-1}(\Omega)$ and $D \subset\subset \Omega_0$. If Assumptions **A.0**, **A.1**, **A.2** and the new **A.3** in Proposition 2.1 hold and $w \in S_h(\Omega_0)$ satisfies*

$$a(w, v) = f(v), \quad \forall v \in H_0^1(\Omega), \quad (3.4)$$

then the following local estimate holds

$$\|w\|_{1,D} \leq C \left(\varepsilon^{\frac{p+1}{2}} h_{\Omega_0}^{-1} \|w\|_{0,\Omega_0} + \sum_{j=0}^p \varepsilon^j (\|f\|_{-1,\Omega_0} + \|w\|_{0,\Omega_0}) \right), \quad (3.5)$$

where C is a constant independent of D and the mesh size, p is the number of mesh layer from D to Ω_0 , ε is defined by

$$\varepsilon = \left(d_{\Omega_0}^{-2} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}} \right)^{2r} + \frac{h_{\Omega_0}}{d_{\Omega_0}} \right)^{\frac{1}{2}}, \quad (3.6)$$

and $\|f\|_{-1, \Omega_0}$ is defined as follows

$$\|f\|_{-1, \Omega_0} = \sup_{\varphi \in H_0^1(\Omega_0), \|\varphi\|_{1, \Omega_0}=1} f(\varphi).$$

Proof. Let p be an integer such that there exist Ω_j ($j = 1, 2, \dots, p$) satisfying

$$D \subset\subset \Omega_p \subset\subset \Omega_{p-1} \subset\subset \dots \subset\subset \Omega_1 \subset\subset \Omega_0.$$

Choose $D_1 \subset \Omega$ satisfying $D \subset\subset D_1 \subset\subset \Omega_p$ and $\omega \in C_0^\infty(\Omega)$ such that $\omega \equiv 1$ on \bar{D}_1 and $\text{supp } \omega \subset\subset \Omega_p$. Then From (2.4), (2.5), (3.4), (3.1) and Assumption A.3, we have

$$\begin{aligned} & \|w\|_{1,D}^2 = \|\omega w\|_{1,D}^2 \leq \|\omega w\|_{1,\Omega_p}^2 \\ & \leq C a_0(\omega w, \omega w) \leq C(a(w, \omega^2 w) + \|w\|_{0,\Omega_p}^2) \\ & \leq C(a(w, \omega^2 w - v) + f(v) + \|w\|_{0,\Omega_p}^2) \\ & \leq C(\|w\|_{1,\Omega_p} \|\omega^2 w - v\|_{1,\Omega_p} + \|f\|_{-1,\Omega_0} \|v\|_{1,\Omega_p} + \|w\|_{0,\Omega_p}^2) \\ & \leq C\left(\|w\|_{1,\Omega_p} \|\omega^2 w - v\|_{1,\Omega_p} + \|\omega^2 w - v\|_{1,\Omega_p} + \|f\|_{-1,\Omega_0} + \|w\|_{0,\Omega_0}\right) \\ & \quad + \frac{1}{2} \|\omega w\|_{1,\Omega_p}^2 \\ & \leq C\left(d_{\Omega_0}^{-1} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}}\right)^r \|w\|_{0,\Omega_p} \|w\|_{1,\Omega_p} + \frac{h_{\Omega_0}}{d_{\Omega_0}} \|w\|_{1,\Omega_p}^2 + \|f\|_{-1,\Omega_0}^2 + \|w\|_{0,\Omega_0}^2\right) \\ & \quad + \frac{1}{2} \|\omega w\|_{1,\Omega_p}^2 \\ & \leq C\left(\left(d_{\Omega_0}^{-2} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}}\right)^{2r} + \frac{h_{\Omega_0}}{d_{\Omega_0}}\right) \|w\|_{1,\Omega_p}^2 + \|f\|_{-1,\Omega_0}^2 + \|w\|_{0,\Omega_0}^2\right) \\ & \quad + \frac{1}{2} \|\omega w\|_{1,\Omega_p}^2. \end{aligned} \quad (3.7)$$

With an application of kick-back argument to (3.7), we have

$$\|w\|_{1,D}^2 \leq C\left(\left(d_{\Omega_0}^{-2} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}}\right)^{2r} + \frac{h_{\Omega_0}}{d_{\Omega_0}}\right) \|w\|_{1,\Omega_p} + \|f\|_{-1,\Omega_0}^2 + \|w\|_{0,\Omega_0}^2\right).$$

Thus, the following estimate holds

$$\|w\|_{1,D} \leq C\left(\left(d_{\Omega_0}^{-2} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}}\right)^{2r} + \frac{h_{\Omega_0}}{d_{\Omega_0}}\right)^{\frac{1}{2}} \|w\|_{1,\Omega_p} + \|f\|_{-1,\Omega_0} + \|w\|_{0,\Omega_0}\right).$$

Note that D can be viewed as Ω_{p+1} , the above process can be taken recursively and leads to

$$\|w\|_{1,\Omega_j} \leq C \left(\left(d_{\Omega_0}^{-2} \left(\frac{h_{\Omega_0}}{d_{\Omega_0}} \right)^{2r} + \frac{h_{\Omega_0}}{d_{\Omega_0}} \right)^{\frac{1}{2}} \|w\|_{1,\Omega_{j-1}} + \|f\|_{-1,\Omega_0} + \|w\|_{0,\Omega_0} \right).$$

Combining the above inequalities and the inverse inequality, we have

$$\begin{aligned} \|w\|_{1,D} &\leq C \left(\varepsilon^{\frac{p+1}{2}} \|w\|_{1,\Omega_0} + \sum_{j=0}^p \varepsilon^j (\|f\|_{-1,\Omega_0} + \|w\|_{0,\Omega_0}) \right) \\ &\leq C \left(\varepsilon^{\frac{p+1}{2}} h_{\Omega_0}^{-1} \|w\|_{0,\Omega_0} + \sum_{j=0}^p \varepsilon^j (\|f\|_{-1,\Omega_0} + \|w\|_{0,\Omega_0}) \right). \end{aligned} \quad (3.8)$$

This is the desired result and the proof is complete. \square

From the delicate local a priori estimate (3.5), we can find that the usual local estimate

$$\|w\|_{1,D} \leq C (\|w\|_{0,\Omega_0} + \|f\|_{-1,\Omega_0})$$

holds only on the case where the scale of Ω_0 is $\mathcal{O}(1)$. It means that the number of subdomains should be $\mathcal{O}(1)$ and the speed up rate of the parallel technique based on local a priori estimate also should be $\mathcal{O}(1)$.

4 Concluding remarks

In this note, we investigate the dependence of the local a priori estimates on the scale of the subdomains. As we know, some domain decomposition and parallel techniques depend on the local a priori estimate of the finite element method. From the derived local estimate (3.5), we can find that the local estimate depends on the scale of the subdomains. This dependence push a constraint to the speed up rate of the parallel technique based on the local error estimate. This is why we consider this problem here and the derived results may give some hints for constructing the domain decomposition techniques to solve partial differential equations.

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